Lecture 6: Discrete & Continuous Probability and Random Variables

D. Alex Hughes

Math Camp

September 17, 2015

Variance of Discrete Random Variables

Definition

The *variance* of a random variable is the expected value of its squared deviations from μ. If X is discrete with pdf $p_X(k)$,

$$VAR[X] = \sigma^2 = E[(X-\mu)^2] = \sum_{\forall k} (k-\mu)^2 \cdot p_X(k).$$

Note that $\mu = E[X]$. Also note that the standard deviation σ is the square root of the variance.

$$VAR[X] = E[(X - E[X])^2]$$

Just as the expected value has a clear analogy to the "center of balance" of a distribution, the variance is analogous to the "moment of inertia."

Imagine placing two distributions on a turntable and pushing on each with identical force 6" from the center. A distribution with low variance would spin faster, and a distribution with high variance would spin slower.

Discrete RV Variance

Example

What is the variance of a six sided die roll?

Discrete RV Variance

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Answer

First calculate the expected value of a single die roll.

$$\sum_{1}^{6} x_i * p(x_i) = 1(1/6) + 2(1/6) + 3(1/6)$$
$$+ 4(1/6) + 5(1/6) + 6(1/6)$$
$$E[X] = \mu = 3.5$$

Discrete RV Variance

Answer

Then the Variance is:

$$\sum_{1}^{6}(x_i - \mu)^2 p(x_i) = (1-3.5)^2(1/6) + (2-3.5)^2(1/6) + (3-3.5)^2(1/6)$$
$$+ (4-3.5)^2(1/6) + (5-3.5)^2(1/6) + (6-3.5)^2(1/6)$$
$$= 17.5(1/6)$$
$$= 2.92$$

Example of Discrete RV Variance

Example

An urn contains five chips, two red and three white. Two chips are drawn at random, *with replacement*. Let X denote the number of red chips drawn. What is VAR[X]?

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Answer

X is binomial, $n = 2$ and $p = \frac{2}{5}$.

$$
\begin{aligned}
E[X] &= np = (2)(2/5) = 4/5 \\
VAR[X] &= \sum (x - E[X])^2 f(x) = \sum_{i=0}^{2} (i - .8)^2 \binom{2}{i} \left(\frac{2}{5}\right)^i \left(\frac{3}{5}\right)^{2-i} \\
&= (0 - .8)^2 \cdot \binom{2}{0} (2/5)^0 (3/5)^2 + (1 - .8)^2 \cdot \binom{2}{1} (2/5)^1 (3/5)^1 + (2 - .8)^2 \binom{2}{2} (2/5)^2 (3/5)^0 \\
&= (16/25)(1)(9/25) + (1/25)(2)(6/25) + (36/25)(1)(4/25) = \mathbf{\frac{12}{25}}
\end{aligned}
$$

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Some process has equal chance of generating a number between [0,1]. What is the probability the process generate a number less that 0.5?

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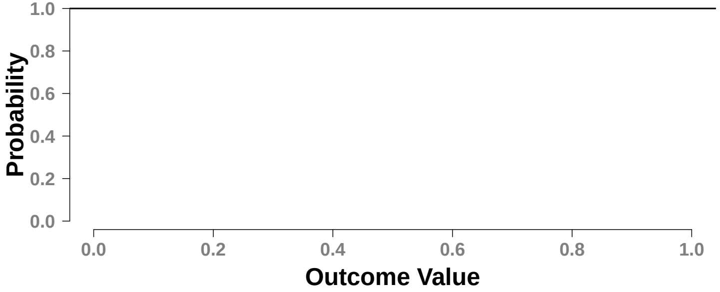

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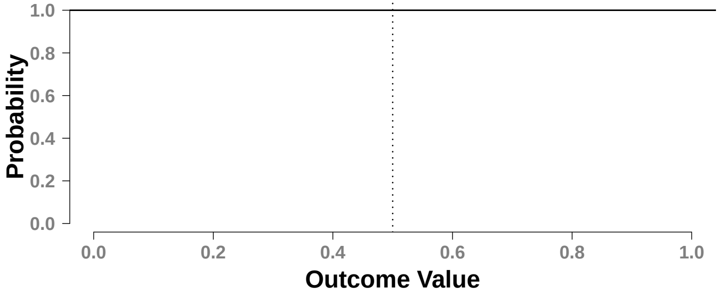

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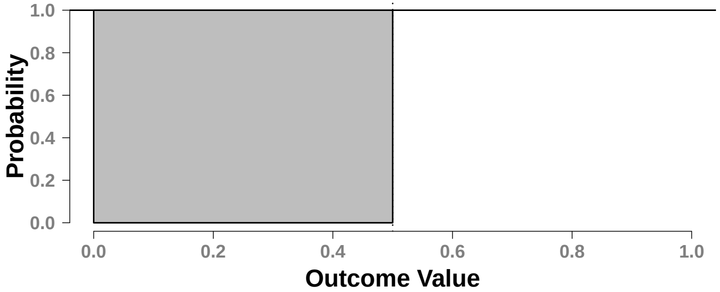

Continuous Probability

Definition

A probability density function P on a set of real numbers S is called *Continuous* is there exists a function f(t) such that for any closed interval [a,b] $\subset$ S,

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Comment

f(t) must satisfy two properties:

1. $f(t) \geq 0$ for all t
2. $\int_{-\infty}^{\infty} f(t)dt = 1$

Frequently, we refer to the probability distribution function as the "pdf."

Continuous Probability Example

Example

- What continuous probability function describes the uniform chance of drawing a number on the range [0,10]?
  - What is the probability that a randomly selected number falls between 4 and 7?
  - What is the probability that a randomly selected number falls between [2,4] or [6,9]?
- What continuous probability function describes the uniform chance of drawing a number on the range [a,b]?

Continuous Probability Example 2

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Answer

- Yes. Recall the two properties f(x) just satisfy: (A) f(x) is always non-negative; (B) The integral of f(x) across the function's domain = 1.
- $\int_0^{1/3} = x^3|_0^{1/3} = \mathbf{1/27}$
- $\int_{2/3}^{1} = x^3|_{2/3}^{1} = 1 - 8/27 = \mathbf{19/27}$.

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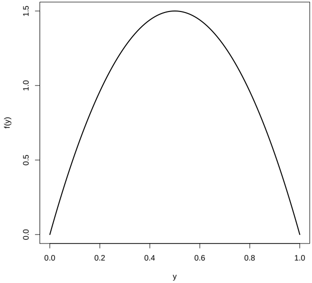

I've drawn one where $f(y) = 6y(1 - y)$.

Example 3

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Does this meet the requirements of a probability density function?

Answer

Yes!

1. $f_Y \geq 0$ for all y;
2. $P(\Omega) = 1$.

Then, with this function, answer the following:

- What is the probability that a number is less than 0.5?
- Greater than 0.5?
- Less than 0.25?
- Greater than 0.25?
- Exactly 0.25?

Cumulative Density Function

Definition

The cumulative density function (called "cdf") of a continuous, real-valued random variable Y is an indefinite integral of its pdf.

$$F_Y(x) = \int_{-\infty}^{x} f_y(x)dx = P(X \leq x)$$

Note the difference in notation between a cumulative distribution function (cdf) and a probability distribution function (pdf).

Theorem

$$\begin{aligned} F(y) &= \int_{-\infty}^{y} f(t)dt; \textit{ conversely,} \\ \frac{d}{dy}F(y) &= f(y) \end{aligned}$$

Cumulative Density Examples

Example

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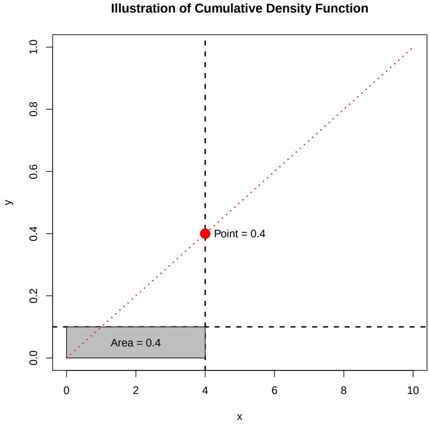

$$\int_0^4 f(x)dx = .1x\Big|_0^4$$
$$= .1(4) - .1(0)$$
$$\int f(x) = F_X(x)$$
$$F_X(x) = 0.1x$$
$$F_X(4) = 0.1(4)$$
$$= .4$$

CDF Properties

Theorem

Let Y be a continuous random variable with cdf $F_Y(y)$, Then,

1. $P(Y > s) = 1 - F_Y(s)$
2. $P(r < Y \leq s) = F_Y(s) - F_Y(r)$
3. $\lim_{x \to +\infty} F_Y(y) = 1$
4. $\lim_{x \to -\infty} F_Y(y) = 0$

Additionally:

- The CDF is a monotonically nondecreasing continuous function.
- Will be useful for easily/quickly calculating the probability a variable takes a value on some interval.

An Example

In Encinitas the distribution of waves a broski, Tad, takes every hour, Y, is described by the pdf:

$$f_Y(y) = y * e^{-y}, y \geq 0.$$

What is the median of Encinitas' wave-income distribution?

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In looking for the median, we are looking for the number where the probability below is equal to the probability above.

$$F_Y(y) = \int_0^\infty ye^{-y}$$
$$= -e^{-y}$$
$$0.5 = -e^{-y}|_y^\infty$$
$$0.5 = 0 - (-e^{-y})$$
$$0.5 = e^{-y}$$
$$ln(0.5) = -y \to \mathbf{0.69}$$

Definition

If Y is a continuous random variable with pdf $f_Y(y)$,

$$E[Y] = \int_{-\infty}^{\infty} y \cdot f_Y(y)dy$$

In the same was as the discrete case, one may think of this as a center of gravity of the distribution.

Example

A continuous uniform density function, f(x) = 1 is defined on the range [0,1]. What is the expected value of this function?

Answer

$$
\begin{aligned}
E[X] &= \int_0^1 x \cdot f(x) dx \\
&= \int_0^1 1x dx \\
&= \frac{1}{2} x^2 |_0^1 \\
&= \frac{1}{2}(1 - 0) \\
&= \mathbf{0.5}
\end{aligned}
$$

Variance of Continuous Random Variables

Definition

Let X be a real-valued random variable with density function $f_X(x)$. Then, the *variance*, σ^2, is defined by

$$\sigma^2 = E[(X-\mu)^2]$$
$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx$$

Expectation Identities

Theorem

For any constant c, $E[c] = c$

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$$E[cX] = cE[X]$$

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For any constant c and random variable X,

$$E[cX] = cE[X]$$

Theorem

For any X and Y that are random variables that have finite expectations

$$E[X + Y] = E[X] + E[Y]$$

Expected Value of a Function of a Random Variable

Theorem

Suppose X is a random variable with pdf $p_X(k)$. Let $g(X)$ be any function of X. Then the expected value of the random variable g(X) is given by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) dx$$

Example

Suppose $X = f_X(x)$ is the uniform density function defined on the range [0,1]. Then $f_X(x) = 1$. Suppose $g(x) = 1/5x$.

$$E[g(X)] = \int_0^1 \frac{1}{5} x \cdot f_X(x) dx$$
$$= \frac{1}{5} [\frac{1}{2} x^2]_0^1 = \mathbf{\frac{1}{10}}$$

Example

The Zetas want to burn their gang-signs into a wall. Suppose the amount of fuel in the torch, Y, is a random variable with pdf

$$f_Y(y) = 3y^2, 0 < y < 1.$$

In the past they have been able to burn a circle whose radius is 20 times the size of y. How much area can they expect to burn?

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Answer

By problem setup, $g(Y) = 20\pi Y^2$

$$E[g(Y)] = \int_0^1 20\pi y^2 \cdot 3y^2 dy = 60\pi \int_0^1 y^4$$
$$= \frac{60\pi y^5}{5}|_0^1$$
$$= 12\pi ft^2$$

Variance

Recall the definition of Variance:

Definition

The *variance* of a random variable is the expected value of its squared deviations from μ. If X is discrete with pdf $p_X(k)$,

$$VAR[X] = \sigma^2 = E[(X-\mu)^2] = \sum_{\forall k}(k-\mu)^2 \cdot p_X(k).$$

Note that $\mu = E[X]$. Also note that the standard deviation σ is the square root of the variance.

$$VAR[X] = E[(X - E[X])^2]$$

A Variance Identity

Theorem

Let X be any random variable (discrete or continuous), having mean μ and for which $E[X^2]$ is finite. Then,

$$VAR[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2$$

Proof.

$$\begin{aligned} VAR[X] &= E[(X - E[X])^2] \\ &= E[X^2 - 2XE[X] + E[X]^2] \\ &= E[X^2] - E[2XE[X]] + E[E[X]^2] \\ &= E[X^2] - E[2X]E[X] + E[X]^2 \\ &= E[X^2] - 2E[X]E[X] + E[X]^2 \\ &= E[X^2] - E[X]^2 \end{aligned}$$

An Example

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Given the following pdf, what is the variance?

$$f(x) = 3(1-x)^2, 0 < x < 1$$

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Answer

$$E[X] = \int_0^1 x \cdot 3(1-x)^2 dx = 3\int_0^1 x - 2y^2 + y^3 = 3$$

$$E[X^2] = \int_0^1 x^2 \cdot 3(1-x)^2 dx = 3\int_0^1 y^2 - 2y^3 + y^4 dx = \frac{1}{10}$$

$$VAR[X] = E[X^2] - E[X]^2 = \frac{3}{80}$$

Variance Identities

Theorem

For and random variable X and constant c

$$VAR[cX] = c^2 VAR[X]$$

$$VAR[X + c] = VAR[X]$$

Proof.

For ease, let $E[X] = \mu$. Then $E[cX] = c\mu$, and

$$\begin{aligned} V[cX] &= E[(cX - c\mu)^2] = E[c^2(X - \mu)^2] \\ &= c^2 E[(X - \mu)^2] \\ &= c^2 VAR[X] \end{aligned}$$

□

Variance Identities

Theorem

Let X and Y be two independent random variables. Then,

$$VAR[X+Y] = VAR[X] + VAR[Y]$$

Proof.

Let E[X] = a and E[Y] = b.

$$\begin{aligned} VAR[X+Y] &= E[(X+Y)^2] - (a+b)^2 \\ &= E[X^2] + 2E[XY] + E[Y^2] - a^2 - 2ab - b^2 \\ &= E[X^2] - a^2 + E[Y^2] - b^2 = VAR[X] + VAR[Y] \end{aligned}$$

□

Introduction

What do we require of a probability distribution/density model?

- It must be non-negative for all outcomes in its sample space
- It must sum or integrate to one across the sample space.

By this definition, $f(y) = \frac{y}{4} + 7\frac{y^3}{2}, 0 \leq y \leq 1$ is a valid pdf.

- $f(y) \geq 0, \forall y 0 \leq 1$, and;
- $\int_0^1 \frac{y}{4} + 7\frac{y^3}{2} dy = 1$

But, how useful is this model?

A Model's Merit

- A pdf has utility as a probability model if it actually models the behavior of the real (or political) world.
- A surprisingly small number of distributions describe these real world outcomes.
- Many measurements/outcomes are the result of the same set of assumptions about the data generating process
    - Number of fraud incidents
    - Number of latrines built
    - Feeding patterns of zebra muscles