270: Discrete Probability

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Math Camp

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Definitions & Theorems

Definitions

- Experiments and Sample Spaces
- Distribution Functions
- Probability
- Sets

Theorems

- Set Theorems
- Two-wise Disjoint Sets
- N-wise Disjoint Sets
- Union of Sets

Another resource

This is another resource that also has all of the .tex code available, if you are so inclined. You can get to it through the authors' homepage.

Sets and Elements of Sets

Definition

- A *set* is a collection of objects. Those things in the set are called the set's *elements*.
  - The collection of objects must be *well-defined*, and
  - The objects must be *distinct*, that is, not counted multiple times.
- By convention, upper-case letters refer to sets, and lower case letters to elements within the set.
- If x is an element of the set S, we say x belongs to S, and write: $x \in S$.

Subsets

Definition

- Imagine that all the elements of set A also belong to set B. That is, $x \in A \rightarrow x \in B$ We would then say that A is a *sub-set* of B, and would write $A \subseteq B$.
- If A and B have exactly the same elements, then A and B are *equal*.

Designating Sets

1. We surround the elements of a set with braces – { ... }
2. We use either a colon (:) or a vertical bar (|) to say "such that"
   For example,
   - SD = {baseball players : the player is on the Padres};
   - Q = {x: a is a Queen}

Set Notation

Definition

- Intersection ($\cap$) – The case that events mutually occur. The logical "AND".
- Union ($\cup$) – The case that events independently occur. The logical "OR".
- Compliment ($\cdot^C$)– The opposite of an event occurring. The logical "NOT".

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Example

- Diagram $A \cap B$
- Diagram $A \cup B$
- Diagram $A \cup B \cap C$
- Diagram $(A \cup B) \cap C$
- Diagram $A \cup (B \cap C)$

Some (More) Set Theorems

Example

Define $A = \{x : 0 \leq x \leq 1\}, B = \{x : 0 \leq x \leq 3\}, C = \{x : -1 \leq x \leq 2\}$.
Draw diagrams showing each of the following points:

1. $A^C \cap B \cap C$
2. $A^C \cup (B \cap C)$
3. $A \cap B \cap C^C$
4. $((A \cup B) \cap C^C)^C$

Sets of Sets and Partitioning Sets

Sets can, themselves, be a part of a set.

Definition

A *class* of sets is a collection of sets. The individual sets can be referenced in just the same way as referencing the elements in a set.

Definition

A Power Set ($\mathcal{P}$) is the class of all proper subsets of a set.

Definition

A *partition* of a set A is a subdivision of A into *non-empty*, *disjoint* subsets, whose union is A. Best to think of this visually.

Example

Consider the classes of $X = \{1,\ 2,\ 3,\ 4,\ 5\}$. Which are partitions?

1. $[\{1, 2\}, \{3\}, \{4\}]$
2. $[\{1, 2, 3\}, \{4, 5\}]$
3. $[\{1, 5\}, \{2\}, \{3, 4\}]$
4. $[\{1, 5\}, \{2\}, \{2, 3, 4\}]$

Partitions

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Answer

1. No. The element 5 is not a member of the class.
2. Yes. All the elements are members of the class.
3. Yes.
4. No. They are not disjoint subsets.

Random Variables and Sample Spaces

Some Preliminaries

Definition

- An *Experiment* is a procedure that:
  1. Can be repeated, theoretically an infinite number of times; and,
  2. Has a well defined set of outcomes.
- A *Sample Outcome* is the result of an experiment. Denoted ω.
- The *Sample Space* is the joint set of all possible sample outcomes. Denoted Ω.
- An *Event* is any collection of size $1, 2, ..., N$ of ω.
- An event *Occurrs* if the outcome of the experiment is is a member of the event.

Three Coin-Flip

Example

Flip a coin 3 times. How many times will the majority of coins show heads?

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- Experiment – Flipping a coin three times
- ω – The ordered Triple
- $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$
- Event – HHH, HHT, HTH, THH

Example

Roll a dice. What is sample space? Let E_1 be the event a number three or lower occurs. Let E_2 be the event an even number occurs. Let E_3 be the event that a 6 is rolled. What are each of the sets? What is $E_1 \cap E_3$? What is $E_1 \cup E_2$? What is $E_1 \cap E_2$?

Baseball

Example

A baseball batter comes to the plate and never swings at a pitch. What outcomes make up the event A that she walks on the sixth pitch?

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Answer

- Walk on the 4th ball
- ω is the ordered 6-ple
- Ω is the set of all ordered n-ples that follow the rules of baseball.
- $A = \{$BBBSSB, BBSSBB, BSSBBB, SSBBB, BBSBSB, BSBSBB, SBSBB, BSBBSB, SBBBSB$\}$

Distribution Function

Definition

A *Distribution Function* $m(\omega_j)$ is a function that assigns a nonnegative number to each sample outcome (ω_j) such that they add to unity.

Formally : A *Distribution Function* for X is a real-valued funciton m whose domain is Ω and which satisfies:

1. $m(\omega) \geq 0, \forall \omega \in \Omega$, and
2. $\sum_{\omega \in \Omega} m(\omega) = 1$

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Example

A fair, six sided die. The chance of getting any particular side should be the same as any other. Since $\sum m(\omega_j) = 1$, then $m(\omega_j) = \frac{1}{6}$.
Why can't $m(\omega_j) = \frac{1}{8}$?

Distribution Function

Definition

The probability of an event E, any subset of Ω is given by:

$$P(E) = \sum_{\omega \in E} m(\omega)$$

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Example

1. What is the probability of drawing the Ace of Spades from a deck of poker cards?
2. What is the probability of drawing an Ace from a deck of cards?
3. What is the probability of drawing a face card from a deck of cards?

Distribution Function

The $m(\cdot)$ need not be uniform in their distribution.

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Example

Let X be a random variable with distribution function $m(\omega)$, where ω is in the set $\{\omega_1, \omega_2, \omega_3\}$, and $m(\omega_1) = 1/2, m(\omega_2) = 1/3$. What then must be $m(\omega_3)$? Which sample outcome will happen more frequently than others? How much more frequently?

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Example

The presidential election, what are the chances that the candidates win the popular vote in each state?

Distribution Function

Example

Example 1.9 Three people, A, B, and C, are running for the same office, and we assume that one and only one of them wins.

What is the sample space?

Suppose A and B have the same chance of winning, but C is a long-shot and has only 1/2 the chance of A or B.

What is the distribution function for each candidate winning? What is the probability, P(E), that either B or C wins?

Distribution Function

Answer

Sample Space, $\Omega = \{A, B, C\}$

To solve for each candidates probability of winning,

$$m(A) = m(B) = 2m(C)$$

$$m(A) + m(B) + m(C) = 2m(C) + 2m(C) + m(C) = 1$$

$$\rightarrow 5m(C) = 1 \rightarrow m(C) = 1/5.$$

$$P(E) = m(B) + m(C) = \frac{2}{5} + \frac{1}{5} = \frac{3}{5}.$$

Set Properties and Probability

Definition

Events A and B are *mutually exclusive* if they have no elements in common. That is, if $A \cap B = \emptyset$.

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Theorem

The probabilities assigned to events by a distribution function on a sample space Ω *satisfy the following properties:*

1. $P(E) \geq 0$ *for every* $E \subset \Omega$.
2. $P(\Omega) = 1$
3. *If* $E \subset F \subset \Omega$, *then,* $P(E) \leq P(F)$.
4. *If* A *and* B *are disjoint subsets of* Ω *then* $P(A \cup B) = P(A) + P(B)$
5. $P(A^C) = P(\Omega) - P(A) = 1 - P(A)$ *for every* $A \subset \Omega$.

Some Set Theorems

Theorem

If $A_1, ..., A_N$ are pairwise disjoint subsets of Ω, then:

$$P(A_1 \cup \cdots \cup A_N) = \sum_{i=1}^{N} P(A_i).$$

Theorem

Let $A_1, ..., A_N$ partition Ω events with $\Omega = A_1 \cup \cdots \cup A_N$, and let E be any event. Then,

$$P(E) = \sum_{i=1}^{N} P(E \cap A_i).$$

Some (More) Set Theorems

Theorem

If A and B are subsets of Ω, then,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Corollary

More generally, for any events A_1, A_2, A_3,

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

Some (More) Examples

Example

Let A and B be events such that:

$$P(A \cap B) = 1/4, P(A^C) = 1/3, P(B) = 1/2.$$

What is $P(A \cup B)$?

Example

Two cars are drawn from a poker deck without replacement. What is the probability that the second is higher in rank than the first?

Example Answer

Answer

Let A_1, A_2, A_3, be the events "First card is lower in rank," "First card is higher in rank," and "Both cards have the same rank," respectively. Each A_i are mutually exclusive, and account for all possible outcomes. Then,

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) = P(\Omega) = 1$$

Once a card is drawn, there are three other cards that can be drawn that would be the same rank, $(A_3) = 3/51$. Before you look at the cards, the second card has an equal chance of being higher as it does lower, so

$$P(A_1) = P(A_2)$$

And so,

$$2P(A_2) + \frac{3}{51} = 1 \rightarrow P(A_2) = \frac{8}{17}.$$

Counting Rules

Hey ma! I'm getting a PhD in Counting!

Theorem

If an operation A can be performed in m ways and B in n ways, then the sequence A,B can be performed in $m \times n$ ways.

Theorem

If k sets, A_k, have n_k mutually distinct elements, then there are

$$\prod_{k=1}^{K} n_k = n_1 \times \cdots \times n_k,$$

ways to arrange those elements.

Example

- You've bet a friend the chance you can pick the champion of all three major sports is better than the chances she picks your drinks, in order, for the night.
  - There are 30 baseball teams, 32 NFL teams, and 29 NBA teams (Lakers don't count).
  - There are 3 types of whiskey (Bourbon, Rye, Scotch) and 6 types of beer (IPA, Pale Ale, Stout, Porter, Sour, Wheat).
  - The bartender has a thing for organization, and you have a thing for booze. He keeps 4 of each type, and you intend to drink one of each type.
- Does your friend buy, or do you owe her?

Permutations

Definition

A permutation is a function (recall: one-to-one mapping) of a set A onto itself. *A permutation is a reordering.*

We can shorten the length of the re-ordering, we just need to account for this.

Theorem

The total number of permutations of set A of n elements is

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n!$$

The total number of k-permutations of set A of n is

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

Permutations

Example

- How many ways can one arrange the three-letter set $\{A, B, C\}$?
- How many ways can one arrange the four-letter set $\{A, B, C, D\}$?
- How many ways can one arrange the 26-letter set of the English alphabet?
- How many ways can one arrange 4-letter "words" from the alphabet?
- How many ways can one arrange a 6 letter word that has two vowels?

Permutations of Repeated Objects

Theorem

The number of ways to arrange n objects, n_1 being of one kind, n_2 of another kind, ..., and n_r of an n^{th} kind, is

$$\frac{n!}{n_1!n_2!...n_r!}; \text{where} \sum_{i=1}^{r} n_i = n$$

Example

Then for the spelling words with two vowels example – How many ways are there to place the vowels and consonants?

Answer

There are $\frac{6!}{2!*4!} = \frac{6*5*4*3*2*1}{(2*1)*(4*3*2*1)} = 15$ ways to arrange the vowels and consonants. Then there are $15 * (21^4 * 5^2)$ words.

Permutations of Repeated Objects

Combinations

- Typically, we don't care about the *order* of a collection of objects. We don't care the order a hand of cards was dealt, or the order a population voted on a candidate.
- For this, we decrease our $n!$ by an appropriate discount factor.

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Theorem

The number of ways to form combinations of size k from a set of n distinct objects, repetitions not allowed, is denoted by the symbol $\binom{n}{k}$ *where*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial Distribution

The choose operator is an important input to the binomial distribution.

Definition

Given n Bernoulli trials each with probability of success p, the *binomial distribution* describes the probability of exactly j successes:

$$B(n,p,j) = \binom{n}{j} p^j q^{n-j}$$

The Binomial distribution describes drawing balls from an urn *with replacement*.

Binomial Distribution

Example

A die is rolled four times. What is the probability we roll exactly three even numbers?

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Answer

$$B(4, 1/2, 3) = \binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{(4-3)} = (4)\left(\frac{1}{8}\right)\left(\frac{1}{2}\right) = 4/16 = 1/4.$$

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A die is rolled four times. What is the probability we roll exactly zero aces?

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Example

A die is rolled four times. What is the probability we roll exactly zero aces?

Answer

$$B(4, 5/6, 0) = \binom{4}{0}\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)^0 = (1)\left(\frac{625}{1296}\right)(1) = 0.48$$

Hypergeometric Distribution

A second distribution describes drawing balls from an urn *without replacement*, also known as sampling.

Theorem

Suppose an urn contains r red chips and w white chips, where $r + w = N$. If n chips are drawn at random, without replacement, and if k is the number of red chips selected, then

$$P(k \text{ red chips are chosen}) = \frac{\binom{r}{k}\binom{w}{n-k}}{\binom{N}{n}}$$

Proof.

First, assume the chips are distinguishable (i.e. numbered). The number of ways to choose red chips is ${}_rP_k$. The number of ways to select white chips is, similarly, ${}_wP_{n-k}$. There are $\binom{n}{k}$ ways to choose where in the sequence the red chips go. □

Proof.

Then, there are $\binom{n}{k}{}_rP_k\ {}_wP_{n-k}$ ways to compose the event of interest. There are ${}_NP_n$ ways to choose n from N *in order*. So, P(k chips are red)

$$
\begin{aligned}
&= \frac{\binom{n}{k} rP_k wP_{n-k}}{{}_NP_n} \\
&= \frac{\frac{n!}{k!(n-k)!}\frac{r!}{(r-k)!}\frac{w!}{(w-n+k)!}}{\frac{N}{(N-n)!}} \\
&= \frac{\frac{r!}{k!(r-k)!}\frac{w!}{(n-k)!(w-n+k)!}}{\frac{N!}{n!(N-n)!}} \\
&= \frac{\binom{r}{k}\binom{w}{n-k}}{\binom{N}{n}}
\end{aligned}
$$

□

Hypergeometric Distribution

Example

Imagine there are 25 potential members of a jury. 23 of them will vote guilty no matter the facts, and 2 will vote innocent no matter the facts. What is the probability of selecting a hung jury?

From Larsen and Marx

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Answer

$$\frac{\binom{2}{1}\binom{23}{11}}{\binom{25}{12}} + \frac{\binom{2}{2}\binom{23}{10}}{\binom{25}{12}} = 0.74$$